On the relative cone conjecture for families of hyperkähler manifolds

(j. w. Andreas Höring & Gianluca Pacienza)

X : normal projective $\mathbb{Q}$-factorial var. $/\mathbb{C}$

§1. Kawamata-Morrison cone conj.

- $N^1(X) :=$ $\mathbb{Z}$-module generated by the classes of Cartier div. / num. equiv.

$$N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

$\rightsquigarrow$ In this $\mathbb{R}$-v.s. we have:

Nef cone

$$\mathrm{Nef}(X) := \overline{\mathrm{Amp}(X)} = \{[D] \mid D \cdot C \geq 0 \ \forall C \text{ curve}\}$$

Effective cone

$\mathrm{Eff}(X) :=$ the convex cone generated by the classes of effective Cartier div.

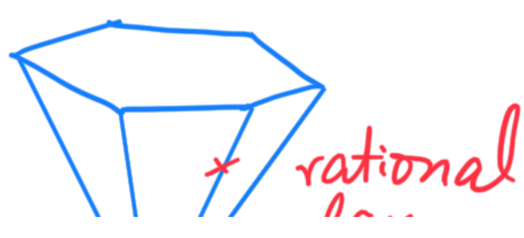
Movable cone

$\overline{\mathrm{Mov}}(X) :=$ the closure of the convex cone generated by the classes of movable div., i.e. effective Cartier div. D s.t. $\mathrm{codim}_X \mathrm{Bs}|D| \geq 2$.

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$$

- If X is Fano,

$\mathrm{Nef}(X)$, $\mathrm{Mov}(X)$, $\mathrm{Eff}(X)$ are all rational polyhedral cones.



- **What if $K_X \equiv 0$?**

Ex:

* Abelian surface of $\rho \geq 3$:

* K3 Kummer: infinitely many (-2)-curves

Kawamata-Morrison Cone Conj

X: normal $\mathbb{Q}$-factorial terminal var. s.t. $K_X \equiv 0$.

(i) Nef cone conj.

$\exists$ a rational polyhedral cone $\Pi \subseteq \mathrm{Nef}(X)$ s.t.

$\mathrm{Aut}(X) \cdot \Pi = \mathrm{Nef}^e(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X)$

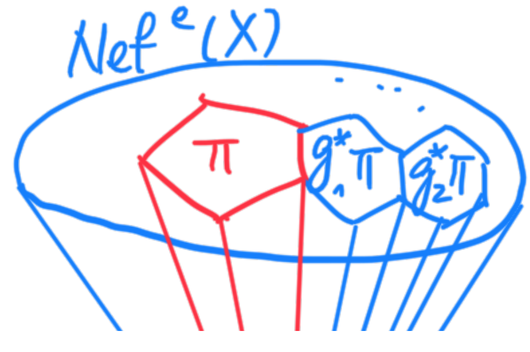
and for every $g \in \mathrm{Aut}(X)$,

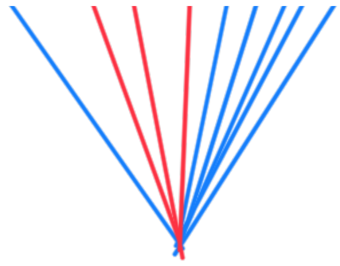
$g^* \mathring{\Pi} \cap \mathring{\Pi} \neq \emptyset$ iff $g^* = \mathrm{Id}$

Π is a fundamental domain of $\mathrm{Nef}^e(X)$ for the action $\mathrm{Aut}(X)$.

(ii) Movable cone conj.

Idem for $\mathrm{PsAut}(X) \circlearrowright \overline{\mathrm{Mov}}^e(X) := \overline{\mathrm{Mov}}(X) \cap \mathrm{Eff}(X)$





Known cases:

(1) Surfaces (Sterk '85, Totaro '10)

(2) Abelian var. (Prendergast-Smith '12)

(3) Hyperkähler mfds
(Markman '11, Amerik-Verbitsky '17)
Sing case: Lehn-Mongardi-Pacienza '23

(4) Some CY3:
Grassi-Morrison, Wilson '93.
Cantat-Oguiso '10,
Hoff-Stenger-Yáñez '22
Gachet-Lin-Wang '23.
....

Observation:
If $\kappa(X) \geq 0$:
MMP predicts: $\exists$ a birat. map $X \dashrightarrow X'$ s.t. $K_{X'}$ is nef
(X' is min. model of X)

abundance conj. $\Rightarrow$ $K_{X'}$ is semiample:

$X' \xrightarrow{\varphi_{|mK_{X'}|}} S$ morphism,
$K_{X'} \equiv_S 0$

$\rightsquigarrow$ we have a family of K-trivial var.

$\rightsquigarrow$ Need a relative version of Cone conj.

§2. Relative cone conj.

Cone conj. (Kawamata '97)

X: normal $\mathbb{Q}$-factorial with terminal sing.
$\pi : X \to S$ fibration over quasi-proj var. S s.t. $K_X \equiv_S 0$

1) (Relative Nef cone conj.)

$\mathrm{Aut}(X/S) \curvearrowright \mathrm{Nef}^e(X/S)$

has a RPFD.

2) (Relative Movable cone conj.)

$\mathrm{PsAut}(X/S) \curvearrowright \overline{\mathrm{Mov}}^e(X/S)$

has a RPFD.

Thm (Cascini - Lazić '14)

Assm: existence of min. models
+ abundance conj.
+ the rel. cone conj.

$\left.\right\}$ in $\dim X$

$\Rightarrow$ If $\kappa(X) \geq 0$, then X has finitely many min. models up to isom.

Known cases:

1) $\dim X = 3$ and $\dim S = 1$ or 2 (Kawamata '97)
2) Relative movable cone conj. when a gen. fib. is K3 (Li - Zhao '22)
3) Relative movable cone conj. in rel. dim at most 2 (Moraga-Stark '24)

Main Thm (HPX '24)

$\pi: X \to S$ fibration between $\mathbb{Q}$-fact. norm. proj. var.

X : klt sing.

Assm : the very general fiber of π is a proj. hyperkähler of one of the 4 known deformation types ($K3^{[n]}$, Kum_n, OG10, OG6)

Then :

(1) The relative movable cone conj. holds for X.

(2) Up to isom. over S, $\exists$ finitely many X' arising as small $\mathbb{Q}$-fact. modification (SQM) of X over S and each X' satisfies the relative nef cone conj.

§3. Subtleties of relative setting.

$\pi: X \to S$ fibration

- D_1, D_2 on X are π-num. equiv. $D_1 \equiv_\pi D_2$ if $D_1 . C = D_2 \cdot C$ $\forall$ $C \subset X$ curve s.t. $\pi(C) = pt$.

$\rightsquigarrow$ $N^1(X/S)$

- A Cartier div. D on X is :

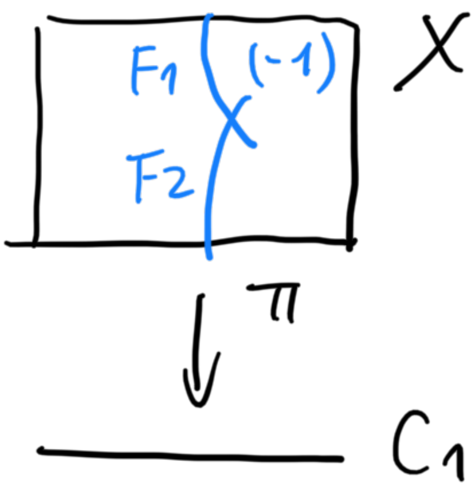
* π-nef if $D . C \geq 0$ $\forall$ $C \subset X$ curve s.t. $\pi(C) = pt$.
* π-effective if $\pi_* \mathcal{O}_X(D) \neq 0$.
* π-movable if

$$\operatorname{codim}\operatorname{supp}\left[\operatorname{coker}\left(\pi^*\pi_*\mathcal{O}_X(D) \xrightarrow[ev]{} \mathcal{O}_X(D)\right)\right] \geq 2.$$

⚠ $\mathrm{Eff}(X/S)$ and $\overline{\mathrm{Mov}}(X/S)$ can be degenerated (i.e. contains a linear subspace of $N^1(X/S)$)

non-deg.   degenerate

Ex: C_1, C_2 curves
$X := Bl_p(C_1 \times C_2)$



$[F_1 + F_2] \equiv 0$ in $N^1(X/C_1)$
$\pi_* \mathcal{O}_X(F_1) \neq 0$

$\Rightarrow [F_i]$ and $-[F_i]$ are both in $\mathrm{Eff}(X/C_1)$

- $\mathrm{Aut}(X/S) = \{ g \in \mathrm{Aut}(X) \mid \pi \circ g = \pi \}$

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

$\mathrm{PsAut}(X/S) = \ \ldots$

[Oguiso '24]: an autom. α_s that exists on every general fiber X_s may not extend to X, even as a birat. automorphism.

(HPX '24) We construct a sm. family of K3 surfaces of Kummer type $\pi : X \to C$ s.t. $\rho(X/S) = \rho(F)$ v.g. fiber $= 17$.

But: $\mathrm{Aut}(X/C)$ is finite,
$(\mathrm{Im}(\mathrm{Aut}(X/C) \to GL(N^1(X/C)))$
$\mathrm{Nef}^e(X/C)$ is rational pol.

§4. Main ingredients

<u>Setup</u>: (X, B) klt pair.
$\pi : X \to S$ K-trivial fibration
i.e. $K_X + B \equiv_\pi 0$

<u>Assm</u>:

① for a v.g. fiber X_s, $q(X_s) = 0$
$\Rightarrow N^1(X/S) \cong \mathrm{Pic}(X/S)$

② S is $\mathbb{Q}$-fact
$\Rightarrow \overline{\mathrm{Mov}}(X/S)$ is non-degenerate
(Li-Zhao '22)

1) Results from MMP:

<u>Thm 1</u> (HPX '24)

Assm good min. models exist for effective klt pairs on X_s
(i.e. for any klt pair (X_s, Δ_s) s.t. $\kappa(K_{X_s} + \Delta_s) \geq 0$, then (X_s, Δ_s) has a good minimal model)

TFAE:

(A) $\mathrm{PsAut}(X/S) \curvearrowright \overline{\mathrm{Mov}}^e(X/S)$ admits a RPFD

(B) (i) For every SQM X' of X

over S,

$$\mathrm{Aut}(X'/S) \circlearrowright \mathrm{Nef}^e(X'/S)$$

admits a RPFD

(ii) Up to isom. over S, $\exists$ finitely many SQMs of X over S.

Rk: (1) $S = \{pt\}$, this is result from: · Gachet – Lin – Stenger – Wang '24
· F, Xu: (A) $\Rightarrow$ (B)

(2) In the relative case, we follow the same strategy, the new input is that we prove a version of "Geography of models" (Shokurov, Choi '11) in the relative setting, based on [Li-Zhao+22].

2) Results on cone conj. for hyperkäh. over field of char 0.

Def: [Takamatsu '21]

K: field of char 0
F: sm. proj. var. over K of dim $2n$.

We say that F is HK if:

- $\pi_1^{\text{ét}}(F_{\bar{K}}) = \{1\}$.
- $H^0(F, \Omega_F^2) \simeq K \cdot \omega_F$.
- $\omega_F^{\wedge n}$ does not vanish.

[Takamatsu '21]:

$$\mathrm{PsAut}(F) \circlearrowright \overline{\mathrm{Mov}}^+(F)$$
$$\shortparallel$$
$$\mathrm{conv}(\overline{\mathrm{Mov}}(F) \cap N^1(F)_{\mathbb{Q}})$$

admits a RPFD.

· Pf of Main Thm : $\pi : X \to S$

$K := \mathbb{C}(S)$

Step 1 : If a v.g. fiber of π is HK, then

X_K is HK over K.

Step 2 : If a v.g. fiber of π is HK s.t. abundance holds, then · $\overline{\mathrm{Mov}}^{+}(X_K) = \overline{\mathrm{Mov}}^{e}(X_K)$.

· $\overline{\mathrm{Mov}}^{+}(X/S) = \overline{\mathrm{Mov}}^{e}(X/S)$.

Step 3 : [Li-Zhao '22]:

Cone conj for $\overline{\mathrm{Mov}}^{e}(X_K)$

$\Rightarrow$ Cone conj. for $\overline{\mathrm{Mov}}^{e}(X/S)$. $\square$